Qualifying Examination in Analysis

January 2014

Please use only one side of the paper and start each problem on a new page. The real numbers are $\mathbb{R}$, the complex numbers are $\mathbb{C}$, and Lebesgue measure on $\mathbb{R}$ is λ.

Please do each problem on separate sheets of paper and ***only use one side of the paper*** (we grade Xerox copies and coping both sides is a pain).

1. Let E be a compact metric space and $\langle f_n \rangle_{n=1}^{\infty}$ a sequence of continuous functions $f_n \colon E \to \mathbb{R}$ such that for all $x \in E$ the sequence $\langle f_n(x) \rangle_{n=1}^{\infty}$ is monotone decreasing and for each $x \in E$ we have $\lim_{n\to\infty} f_n(x) = 0$. Prove

$$\lim_{n\to\infty} f_n = 0$$

uniformly.

2. Let $f \colon \mathbb{C} \to \mathbb{C}$ be an entire function that satisfies

$$|f(z)| \le Ce^{\operatorname{Re} z^2}$$

for some positive constant C. Prove

$$f(z) = ae^{z^2}$$

for some constant $a \in \mathbb{C}$.

3. Compute

$$\int_{|z-1|=1} \frac{e^z \, dz}{z^4 - 1}.$$

4. Find all continuous functions $f \colon [0,1] \to \mathbb{R}$ such that

$$\int_0^1 f(x)x^n \, dx = \frac{1}{(n+1)(n+2)} \quad \text{for} \quad n = 0, 1, 2, 3, \ldots$$

Hint: One such function is $f(x) = 1 - x$.

5. Let E be a measurable subset of $[0,1] \times [0,1]$. For $x \in [0,1]$ let

$$E_x = \{y : (x,y) \in E\}.$$

If

$$\lambda\Big\{x : \lambda(E_x) \ge \frac{1}{2}\Big\} \ge \frac{3}{4}$$

show

$$\lambda \times \lambda(E) \ge \frac{3}{8}$$

and give an example to show this lower bound is the best possible.

6. Prove Fatou's Lemma: If $\{f_n\}$ is a sequence of nonnegative measurable functions and $f_n(x) \to f(x)$ almost everywhere on a set E, then

$$\int_E f \le \liminf \int_E f_n.$$

7. Let $f \in L^1(\mathbb{R})$ and let f_n be the function

$$f_n(x) = f(x - 1/n).$$

Prove

$$\lim_{n\to\infty} \|f - f_n\|_{L^1} = 0.$$

8. Let $f \in L^1(\mathbb{R})$ such that for all closed bounded intervals I

$$\int_I f \, d\lambda = 0.$$

Show $f = 0$ almost everywhere.

9. Let f be increasing on $[0, 1]$ and

$$\int_0^1 f' = f(1) - f(0).$$

Prove that f is absolutely continuous on $[0, 1]$.

10. Prove or give a counterexample.

(a) If K is a compact subset of the irrational numbers, then K has measure zero.

(b) Let $f_n \in L^1([0, 1])$ with $\|f_n\|_{L^1} \le 1/n^2$. Then $\lim_{n\to\infty} f_n(x) = 0$ for almost all $x \in [0, 1]$.

(c) There is an entire function $f(z)$ such that $f(n) = 0$ for all integers n.

(d) There is a non-constant function $f\colon [0, 1] \to \mathbb{R}$ with

$$|f(x) - f(y)| \le 42|x - y|$$

for all $x, y \in [0, 1]$ and $f'(x) = 0$ for almost all $x \in [0, 1]$.

(e) There is a function, f, analytic on the unit disk $D := \{z : |z| < 1\}$ with $|f(z)| \le 3$ for all $z \in D$, $f(0) = 0$ and $f'(0) = 4$.